WNE Linear Algebra
Resit Exam
Series B

24 February 2023

Please use separate sheets for different problems. Please use single sheet for all questions. <u>Please include all relevant calculations</u>. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks. Each question is worth 4 marks.

Problems

Problem 1.
Let $V = \text{lin}((1, 2, 9, 12), (1, 3, 12, 18), (-1, 1, 0, 6))$ be a subspace of $\mathbb{R}^4$.

a) find a basis and the dimension of the subspace V,
b) find a system of linear equations which set of solutions is equal to V.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + 2x_2 + 10x_3 + 6x_4 = 0 \\ x_1 + x_2 + 6x_3 + 5x_4 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ and the dimension of the subspace V,
b) let $v = (0, -7, -2t, t) \in \mathbb{R}^4$. Find all $t \in \mathbb{R}$ such that $v \in V$ (i.e., vector v belongs to V) and compute coordinates of v relative to $\mathcal{A}$.

Problem 3.
Let $\mathcal{A} = ((1, 0), (1, -1))$ be a basis of $\mathbb{R}^2$ and let $\mathcal{B} = ((1, 0, 0), (0, 1, 1), (-1, 0, 1))$ be a basis of $\mathbb{R}^3$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by the matrix

$$M(\varphi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}.$$

Let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by the formula

$$\psi((x_1, x_2)) = (x_2, x_1 + x_2).$$

a) find the formula of φ.
b) find the matrix $M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}}$.

Problem 4.
Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - 2x_2 + x_3 = 0\}$ be a subspace of $\mathbb{R}^3$.

a) find an orthonormal basis of $V^\perp$,
b) compute the orthogonal reflection/symmetry of $w = (4, 1, 3)$ across V.

Problem 5. a) let $q\colon \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form given by the formula

$$q((x_1, x_2)) = -2x_1^2 + 6x_1x_2 - 5x_2^2.$$

Check if the form q is negative definite.

b) let $Q\colon \mathbb{R}^3 \to \mathbb{R}$ be a quadratic form given by the formula

$$Q((x_1, x_2, x_3)) = x_1^2 + 5x_2^2 + 4x_3^2 + 4x_1x_3.$$

Check if the form Q is positive semidefinite.

Problem 6.
Consider the following linear programming problem $x_2 - x_5 \to \min$ in the standard form with constraints

$$\begin{cases} -\; x_1 + x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + x_4 = 2 \\ 3x_1 - 3x_2 + x_5 = 6 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 5.$$

a) which of the sets $\mathcal{B}_1 = \{1,2,4\},\ \mathcal{B}_2 = \{2,3,5\},\ \mathcal{B}_3 = \{3,4,5\}$ is basic feasible? write the corresponding basic solution for all basic sets,
b) solve the linear programming problem using simplex method.

Questions

Question 1.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be matrices such that $A^2 = A,\ B^2 = B$ and $AB = BA$. Does it follow that

$$(A + B + AB)^2 = A + B + 7AB?$$

Solution 1.
Yes, it does.

$$(A+B+AB)^2 = (A+B+AB)(A+B+AB) = A(A+B+AB)+B(A+B+AB)+AB(A+B+AB) =$$
$$= A^2 + AB + A^2B + BA + B^2 + BAB + ABA + AB^2 + ABAB =$$
$$= A + AB + AB + AB + B + AB + AB + AB + AB = A + B + 7AB.$$

Question 2.
Let $A = [a_{ij}] \in M(3 \times 3; \mathbb{R})$ be a matrix. Let

$$B = \begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix}.$$

Does it follow that $\det A = \det B$?

Solution 2.
Yes, it does. By the Sarrus rule $\det B = \det A$, that is all minus signs cancel.

Question 3.
Give an example of a matrix $A \in M(2 \times 2; \mathbb{R})$ with eigenvalues $\lambda = 0$ and $\lambda = 2$ such that

$$V_{(0)} = \operatorname{lin}((1,-1)), \quad V_{(2)} = \operatorname{lin}((1,0)).$$

Is matrix A uniquely determined?

Solution 3.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The conditions are equivalent to

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

that is

$$\begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This gives $a = 2,\ c = 0$, and consequently $b = a = 2$ and $d = c = 0$. Using non–zero vectors proportional to $(1,-1)$ and $(1,0)$, that is $(p,-p)$ and $(q,0)$ for $p, q \neq 0$ yields the same numbers, therefore matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ is uniquely determined.

Question 4.
Let $A \in M(2 \times 2; \mathbb{R})$ be an antisymmetric matrix, i.e. $A^\intercal = -A$. Does it follow that for any $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

$$(Av) \cdot v = 0,$$

where $v \cdot w$ denotes the standard scalar product of vectors $v, w \in \mathbb{R}^2$?

Solution 4.

Yes, it does. If $A = -A^\intercal = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, then

$$(Av) \cdot v = \begin{bmatrix} av_2 & -av_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = av_1v_2 - av_1v_2 = 0.$$

In general, for any matrix $A = -A^\intercal \in M(n \times n; \mathbb{R})$ and $v \in \mathbb{R}^n$

$$(Av) \cdot v = (Av)^\intercal v = v^\intercal A^\intercal v = -v^\intercal (Av) = -v \cdot (Av),$$

that is

$$2(Av) \cdot v = 0 \Longrightarrow (Av) \cdot v = 0.$$

Question 5.

Give an example of a formula of an indefinite quadratic form $q \colon \mathbb{R}^2 \to \mathbb{R}$ such that $q((1,0)) = -10,\ q((0,1)) = -11$ or prove that it does not exist.

Solution 5.

Let $q((x_1, x_2)) = ax_1^2 + bx_1x_2 + cx_2^2$. Since $q((1,0)) = a$ and $q((0,1)) = b$, we have

$$q((x_1, x_2)) = -10x_1^2 + bx_1x_2 - 11x_2^2.$$

Indefinite quadratic form attains positive and negative values. We have, for example

$$q((1,1)) = -21 + b,$$

hence it is enough to take, for example, $b = 22$. Finally

$$q((x_1, x_2)) = -10x_1^2 + 22x_1x_2 - 11x_2^2.$$